Deviations Due to Non-Newtonian Influences Within a Miniature Viscous Disk Pump

A miniature viscous disk pump (VDP) is utilized to characterize and quantify non-Newtonian fluid deviations due to non-Newtonian influences relative to Newtonian flow behavior. Such deviations from Newtonian behavior are induced by adding different concentrations of sucrose to purified water, with increasing non-Newtonian characteristics as sucrose concentration increases from 0% (pure water) to 10% by mass. The VDP consists of a 10.16 mm diameter disk that rotates above a C-shaped channel with inner and outer radii of 1.19 mm, and 2.38 mm, respectively, and a channel depth of 200 μm. Fluid inlet and outlet ports are located at the ends of the C-shaped channel. Within the present study, experimental data are given for rotational speeds of 1200–2500 rpm, fluid viscosities of 0.001–0.00134 Pa s, pressure rises of 0–220 Pa, and flow rates up to approximately 0.00000005 m/s. The theory of Flümerfeldt is modified and adapted for application to the present VDP environment. Included is a new development of expressions for dimensionless volumetric flow rate, and normalized local circumferential velocity for Newtonian and non-Newtonian fluid flows. To quantify deviations due to the magnitude non-Newtonian flow influences, a new pressure rise parameter is employed, which represents the dimensional pressure rise change at a particular flow rate and sucrose concentration, as the flow changes from Newtonian to non-Newtonian behavior. For 5% and 10% sucrose solutions at rotational speeds of 1200–2500 rpm, this parameter increases as the disk dimensional rotational speed increases and as the volumetric flow rate decreases. Associated magnitudes of the pressure difference parameter show that the fluid with the larger sucrose concentration (by mass) produces significantly larger differences between Newtonian and non-Newtonian fluid flow, for each value of dimensional volumetric flow rate. For each disc rotational speed, compared to Newtonian data, dimensional pressure rise variations with dimensional volumetric flow rate, which are associated with the non-Newtonian data, are generally lower when compared at a particular volumetric flow rate. Agreement with analytic results, for any given flow rate, rotational speed, and flow passage height, validates the shear stress model employed to represent non-Newtonian behavior, as well as the analytic equations and tools (based upon the Navier–Stokes equations) which are employed to predict measured behavior over the investigated range of experimental conditions. [DOI: 10.1115/1.4023408]

Introduction

Many different types of micropumps are utilized to induce fluid motion through millimeter-scale or microscale channels in applications related to devices, such as microsensors, separation devices, drug delivery systems, and electronics cooling systems [1]. The variety of pumps employed include membrane pumps [2–8] (both without check valves [2–5] and with check valves [6–8]), electrohydrodynamic (EHD) pumps [9–11], electrokinetic (EK) pumps [12,13], viscous pumps [14,15], rotary pumps [16,17], peristaltic pumps [18–20], ultrasonic pumps [21,22], and several other types of pumps [23–26]. Many of these micropumps are fabricated using microfabrication technology. Nonmechanical pumps like the electrohydrodynamic and electrokinetic pumps do not have moving parts, which increases reliability. However, such devices are generally limited by low flow rate and pressure rise capabilities, the applications of the pump, the working fluids that can be pumped, and high supply voltage requirements [1]. Mechanical pumps like rotary pumps, peristaltic pumps, and membrane pumps have a wide variety of possible working fluids and applications. However, such mechanical micropumps are believed to be feasible only when they are greater than a certain size [1], due to the large viscous forces in the fluid at small pump geometries. At very small scales, the viscous forces are significant, and result in large pressure drops over a small length for fluid flow through a channel [27]. Different uses of different viscous pumps at microscales are described by Sen et al. [15], and Kilani et al. [14]. Of these two studies, Sen et al. [15] describe a pump that employs a shaft, whose axis is perpendicular to the flow direction, and is positioned eccentrically in a channel. The difference in viscous shear between the shaft and the two channel walls produces a net pumping effect. Kilani et al. [14] present information regarding a spiral pump, which uses one spinning disk rotating over a single spiral channel to produce a pumping effect. Results from a macroscale version of this pump are consistent with an analytic expression for flow rate and pressure rise [14]. Other types of macroscale viscous pumps are described in the archival literature [28–35]. Most of these pumps have a linear relationship between...
Within the present investigation, a miniature viscous disk pump (VDP) of millimeter-scale, with a channel depth of 200 μm, is utilized to characterize and quantify deviations of non-Newtonian fluid behavior relative to Newtonian flow behavior. Such deviations from Newtonian behavior are induced by adding different concentrations of sucrose to purified water, with increasing non-Newtonian characteristics as sucrose concentration increases from 0% (pure water) to 10%, by mass. As such, experimental data are given for rotational speeds of 1200–2500 rpm, fluid viscosities of 0.001–0.00134 Pa s, pressure rises of 0–220 Pa, and flow rates up to approximately 0.00000005 m³/s. The theory of Flumerfelt et al. [42] is modified and adapted for application to the present VDP environment, including new development of expressions for dimensionless volumetric flow rate, and normalized local circumferential velocity for non-Newtonian fluid flows. To quantify deviations due to the magnitude non-Newtonian flow influences, a new pressure rise parameter is employed, which represents the dimensionless pressure rise change at a particular flow rate and sucrose concentration, as the flow changes from Newtonian to non-Newtonian behavior. Comparisons of experimental data are presented, which show that fluids with the larger sucrose concentration (by mass) produce significantly larger differences between Newtonian and non-Newtonian fluid flow, for each value of dimensionless volumetric flow rate. Comparisons are also made between experimental data and analytical results, wherein the latter are based upon Navier–Stokes equation derivations, and a power-law shear stress/strain rate model.

Miniature Viscous Disk Pump Configuration and Operation

The pump consists of a spinning disk and a C-shaped channel with a fluid inlet port and a fluid outlet port, which are located at opposite ends of the C-shaped flow channel. The overall arrangement is shown in Figs. 1 and 2, where a coordinate system is also included in the latter figure. The C-shaped channel is milled into a thin sheet of stainless steel and mounted to a block, as illustrated by the photograph which is presented in Fig. 3. This allows for the fluid chamber height, the distance between the disk and the bottom of the channel, to be readily adjusted to preset heights simply by changing channel pieces. The present working channel height is 200 μm. As the disk rotates, its edges remain in contact with the...
fluid chamber walls, creating a seal that minimizes leakage of the test fluid from the fluid chamber.

As the disk spins, a rotating Couette-type flow is induced in the fluid chamber between the rotating disk and the stationary bottom of the channel. A circumferential pressure gradient is then present in the fluid chamber because of the work done on the fluid within the C-shaped channel by the rotating disk. This interaction gives a static pressure rise with circumferential position through the pump chamber volume, such that a region of lower static pressure is present near the fluid inlet port, and a region of higher static pressure is present near the fluid outlet port. This static pressure variation then opposes the motion induced by the disk rotation and viscous forces. If the opposing circumferential static pressure variation is large enough, some of the fluid between the spinning disk and the top surface of the fluid chamber recirculates in a direction which is opposite to the direction of the disk rotation. For the situation when the fluid outlet port is closed (or a valve on the outlet tubing is closed), the overall volumetric flow rate is zero. However, fluid movement continues to be present within the pump chamber in directions which are coincident with and also opposed to the direction of disk rotation. With this arrangement, the resulting static pressure rise is maximum for a particular disk rotational speed.

Pump Component Fabrication

There are three main components of the pump: (i) the fluid chamber assembly, (ii) the disk and shaft, and (iii) the motor and pump housing.

The fluid chamber assembly is comprised of a steel block, a thin channel piece, and a clamp piece. The steel block is machined from type 3-16 stainless steel. All sides are square and the pressure ports and fluid inlet and outlet ports are drilled through the block. The holes for the pressure ports are 0.19 mm in diameter while holes for the fluid inlet and outlet ports are 1.40 mm in diameter. The pressure port holes are widened below the surface of the block to 1.59 mm. The inlet and outlet ports maintain the same diameter through the block, so that plastic adapters are press fit into the back to connect the tubing. There are four M2 threaded holes located at the corners of the block so the channel piece and clamp may be tightly secured to the block. The channel pieces are also constructed from type 3-16 stainless steel. The thickness, which corresponds directly to channel height, is 200 \( \mu \)m. The channel pieces are 2.54 cm \( \times \) 2.54 cm square segments, with C-shaped channel shaped cut from central regions. The C-shaped channel has an inner radius of 1.19 mm and an outer radius of 2.38 mm. The clamp piece is machined from aluminum, also with 2.54 cm \( \times \) 2.54 cm square dimensions, and 1.59 mm thickness. There is a 12.7 mm diameter hole cut from the center to allow the disk to contact the channel piece.

The disk is machined from PEEK plastic. A lathe is used to obtain the desired diameter and length. The diameter of the disk is 10 mm. A hole is cut into one side of the disk so it may slide onto the shaft. It is secured to the shaft with two small set screws.

When assembled, the aluminum motor and pump housings have the shape of an “L,” as shown in Fig. 4(a). On the horizontal portion of the “L,” there is a linear slide. Mounted to the linear slide are two aluminum blocks. The motor is mounted to the outer block and the shaft runs through the inner block. Between the blocks, there is a coupling to absorb any vibrations between the motor and the shaft. The vertical portion of the “L” housing contains the pump assembly. There is a square shaped opening cut into the top portion of the “L” shaped housing. A flat bar sits over the opening and is screwed down to clamp the chamber assembly in place. This ensures that the fluid chamber assembly is perfectly perpendicular to the “L” shaped housing. The linear slide and fluid chamber assembly are thus mounted on the “L” housing to ensure that the disk maintains uniform spacing with respect to the surface of the chamber assembly as testing is underway.

Experimental Apparatus and Procedures

The disk pump is powered by an externally mounted Maxon EC32 #118891, brushless DC motor. The power rating of the motor is 80 W, with a maximum speed of 25000 rpm, and a stall torque of 0.35 N m. The motor is controlled by an Advanced Motion Controls power amplifier, model No. BE12A6. The power amplifier has a dc supply voltage of 48 V, a peak current of 12 A, and a continuous current rating of 6 A. A negative feedback controller is employed to maintain constant speed for any variation in torque. The motor rotational speed is controlled by adjusting a 15-turn potentiometer, and read using a Maxon HEDS 55 number 110513 encoder. In the present investigation, the rotational speed is varied from 250 to 2500 rpm. The speed is displayed in real-time by means of a Red Lion Controls PAXI display. The rotational speed measurement system is calibrated using a timing light.

A DP-15 Validyne differential pressure transducer is used to measure the differential pressure between the pressure ports. The transducer employs a number 20 diaphragm which can measure pressure differentials with ranges up to 872 Pa. The output signal from the pressure sensor is processed using a Celeesco Model number CD15 Carrier Demodulator. This demodulator produces a voltage output that is read by National Instruments USB-6210 data acquisition board and LABVIEW 32-bit software, version 10.0.1. A Key Instruments model 1XLW9 flow meter is used to measure the fluid flow. All data are recorded, and then entered and processed using a Dell Precision T3500 computer workstation.

The pump assembly is mounted to the base of a linear slide. The brushless motor and disk shaft are also mounted to the shuttle of the linear slide, as shown in Figs. 4(a) and 4(b). The disk shaft is supported with two bearings, and the distal end of the disk shaft connects to the motor shaft. Elastic bands are employed to exert a constant force on the shuttle in the direction of the pump chamber to keep the disk surface flush against the bottom of the pump housing. An alignment fixture is used to mount the pump housing and pump chamber, such that the axis of the disk aligns with the center point of the pump chamber radius. The pump assembly is placed on a level surface to ensure that there are no height differences between the flow inlet and outlet ports or between the two pressure ports. After assembling the disk, disk shaft, and pump chamber, the fluid inlet and outlet tubing and pressure port tubing are press-fix into the top of the pump housing. Before the test start, the tubing, flow meter, and pump housing are filled with the working fluid. Syringes are used to remove any air from the liquid which is utilized within the system.

The working fluids are pure water, and 1%, 5%, and 10% sucrose solutions. Associated magnitudes of absolute viscosity and the stress/strain rate power law parameter are given in Table 1. Magnitudes of the power law parameters are determined to provide the best matches between analytic predictions and
experimental data. After the facility assembly steps, mentioned above, are completed, testing is comprised of the following procedures: (1) The motor is activated, the pump is run for 5 min, and then the valve in the flow meter is closed. (2) Once the pressure reading is steady, pressure rise is recorded using LABVIEW 32-bit software, version 10.0.1, and the flow rate is recorded, as measured using the rotameter flow meter. (3) The valve is then opened slightly (at approximately 20% of the full open position). (4) Steps (2) and (3) are repeated about five times until the valve is fully opened, with subsequent data recordings for each flow rate.

**Experimental Uncertainty Estimates**

A first order uncertainty analysis is performed using a constant-odds combination method, based on a 95% confidence level as described by Moffat [45]. The flow passage height variation across the fluid chamber is less than 1.5 mm, with a measured variation less than 1.1 mm (± 0.55 mm) for \( h = 200 \) mm, which is 0.55% of \( h \). The value 0.55% is the maximum percent variation of flow passage height for the flow passage height tested. The resulting uncertainty magnitudes associated with experimentally measured pressure rise, flow passage height, fluid viscosity, disk rotational speed, pump chamber radii, and flow rate are presented in Table 2.
Non-Newtonian Flow Analysis for the Miniature Viscous Disk Pump

Figure 2 shows a cross-sectional view of the VDP flow passage [37]. The flow analysis is applied to the shaded region within this figure. To develop an expression for pressure rise and flow rate in terms of pump geometry, disk rotational speed, and fluid properties, a number of assumptions regarding the fluid behavior, fluid properties, and flow passage configuration are implemented. First, an incompressible, steady flow of a Newtonian fluid, with constant density and viscosity is considered. Assuming that \( h \ll (R_2-R_1) \), edge effects near the inner radius and outer radius can be ignored. Also assume that gravity is negligible, and the rotational speed of the disks is slow enough such that the body force due to centrifugal acceleration is negligible compared to the forces produced by gradients of viscous stresses, and gradients of pressure, an assumption which is valid for low Reynolds number flows. The gradients of viscous stresses are assumed to be more significant than the inertial or advection terms in the Navier–Stokes equation because the fluid motion is produced by viscous forces, and because the flow passage height \( h \) is relatively small [37].

For a cylindrical coordinate system applied to the shaded flow area in Fig. 2, the velocity \( v_\theta \gg v_r \), so the \( z \)-component of velocity at all locations can be approximated as \( v_z=0 \). Also, if it is assumed that \( h \ll R_1 \pi/2 \) (circumferential extent at the inner radius of the pump chamber), the flow in this region can be approximated as fully developed in the \( \theta \)-direction. Assuming incompressible, steady, fully developed flow, and the continuity equation becomes

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r v_r \right) = 0
\]

From this equation, the value of \( v_r \) is equal to a constant. At \( r=R_1 \) (inner radius wall), \( v_r=0 \), which implies that \( v_r=0 \) in the entire region is considered. The pressure gradients for the \( z \)- and \( r \)-directions are then given by \( \partial p/\partial z = 0 \) and \( \partial p/\partial r = 0 \), respectively, which means that the pressure gradient is invariant in the \( z \)-direction and in the \( r \)-direction [37].

Because the flow passage height \( h \) is small, \( h/(R_2-R_1) \ll 1 \), and changes of local circumferential velocity \( v_\theta \) across the \( z \)-direction over the distance \( h \) are much larger than the changes of \( v_\theta \) in the \( r \)-direction over the distance \( R_2-R_1 \). With these considerations, according to Blanchard et al. [37], the resulting Navier–Stokes equation is then given by

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r v_\theta \right) = \mu \frac{\partial^2 v_\theta}{\partial z^2}
\]  

If either a Newtonian or non-Newtonian fluid flow is considered, Eq. (1) can be rewritten in more general form, as follows:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r v_\theta \right) = \frac{\partial \tau}{\partial z} \tag{2}
\]

Because the flow is fully-developed and incompressible, the circumferential velocity \( v_\theta \) does not change with circumferential position. Thus, because mass must be conserved, the pressure gradient on the left-hand sides of Eqs. (1) and (2) are constant in the \( \theta \)-direction, which results in a linear pressure rise in the \( \theta \)-direction. As a result, the pressure gradient in the \( \theta \)-direction is then approximated using

\[
\frac{\partial p}{\partial \theta} \approx \frac{\Delta p}{\Delta \theta} \tag{3}
\]

where \( \Delta p \) is the pressure rise over the circumferential angle \( \Delta \theta \). The pressure rise between the two pressure ports \((p_2-p_1)\) shown in Fig. 2 is denoted as \( \Delta p \), and the pressure rise between the fluid inlet and outlet ports shown in Fig. 2 is denoted as \( \Delta p_{\text{out-in}} \). Also, \( \Delta \theta \) is the angle between the pressure ports \((\Delta \theta = \pi/2)\) or between the inlet and outlet ports \((\Delta \theta = 1.067\pi)\), as shown in Fig. 2.

Following Flumerfelt et al. [42], Eq. (2) is applied along a mean radius line. Combining with Eq. (3) then gives

\[
\frac{1}{(R_1+R_2)/2} \frac{\Delta p}{\Delta \theta} \frac{\partial \tau}{\partial z} = \frac{\partial p}{\partial \theta} \tag{4}
\]

where \( \tau = -h \frac{\Delta p}{\Delta \theta} (\lambda - \xi) \) shown in Fig. 2 is denoted as \( \tau \), and \( \xi = z/h \). For Newtonian and non-Newtonian fluid flows, the local shear stress \( \tau \) can then be given by an equation of the form [42]

\[
\tau = \mu \left( \frac{d\phi}{d\xi} \right)^n \tag{5}
\]

where \( \mu \) is the effective absolute viscosity, and \( n \) is the associated power-law parameter. Combining Eq. (7), and then Eq. (8) gives

\[
\frac{d\phi}{d\xi} = -h \left( \frac{\Delta p}{\Delta \theta} \right) \left[ \frac{\Delta \phi}{\Delta \xi} \right]^{\lambda - \xi} = \Lambda (\lambda - \xi) \tag{9}
\]

where

\[
\Lambda = -h \left( \frac{\Delta p}{\Delta \theta} \right) \left[ \frac{\Delta \phi}{\Delta \xi} \right]^{\lambda - \xi} \tag{10}
\]

Within Eqs. (7)–(10), \( \Lambda \) and \( \lambda \) may be positive or negative. The rightmost part of Eq. (9) is rearranged to become the equation which, according to Ref. [42], is given by

\[
\frac{d\phi}{d\xi} = [\Lambda (\lambda - \xi)]^{1/n} d\xi \tag{11}
\]

where \( s = 1/n \). Integrating this equation and applying nondimensional forms of the boundary conditions given by Eqs. (5) and (6) subsequently produces an equation for the normalized local circumferential velocity at mean radius location \((R_1+R_2)/2\). According to Flumerfelt et al. [42], this normalized local circumferential velocity equation is subsequently given by

Table 2 Experimental uncertainties associated with experimental data

<table>
<thead>
<tr>
<th>Variable</th>
<th>Maximum percent uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta p )</td>
<td>5%</td>
</tr>
<tr>
<td>( Q )</td>
<td>2.50%</td>
</tr>
<tr>
<td>( h )</td>
<td>2.75%</td>
</tr>
<tr>
<td>( \mu )</td>
<td>2%</td>
</tr>
<tr>
<td>( \Omega )</td>
<td>1.50%</td>
</tr>
<tr>
<td>( R_1 ), ( R_2 )</td>
<td>1.10%</td>
</tr>
</tbody>
</table>
\[ \varphi = \int_{-\frac{1}{2}}^{\lambda} \Lambda'(\lambda - \xi) d\xi = \frac{1}{\Lambda(s+1)} \left[ \Lambda\left(\lambda + \frac{1}{2}\right)^{s+1} - \Lambda(\lambda - \xi)^{s+1} \right] \] (12)

This result is applicable to Newtonian and non-Newtonian fluid flows, provided the relationship between stress and strain rate is expressed using Eq. (8).

The result given by Eq. (12) is equivalent to the dimensional velocity equation from Blanchard et al. [37] for Newtonian fluid flows in a microscopic VDP, which is expressed using

\[ v_0(r, z) = \frac{h^5}{2\mu} \left( \frac{1}{h} - \frac{z}{h} \right) + \frac{\rho(r^2 - h^2)}{4} \] (13)

Important differences in the forms of these two equations (for Newtonian fluid flows only) are due to different origins for the \( z \)-coordinate systems, and the applicability of Eq. (12) only along the mean radius of the device.

From Blanchard et al. [37], the volumetric flow rate is determined by integrating the velocity profile given by Eq. (13) over a radial cross section of the pump chamber, from \( z = 0 \) to \( z = h \), and from \( r = R_2 \) to \( r = R_1 \), as given by

\[ Q = \int_{R_1}^{R_2} v_0(r, z) dr dz = \frac{h^3}{12\mu} \frac{\Delta P}{\Delta \theta} + \frac{c_0 h(R_2^2 - R_1^2)}{4} \] (14)

Equation (14) describes Newtonian flow of the single-VDP as dependent upon pressure rise (\( \Delta P \)), pump geometry (\( R_1, R_2, \Delta \theta \), and \( h \)), disk angular velocity (\( \omega \)), and fluid viscosity (\( \mu \)).

From the theoretical development of Flumerfelt et al. [42], the dimensionless volumetric flow rate is expressed using \( \Omega = Q/(WhV) \), where \( W \) is the width of the flow passage in the radial direction (given on a per unit basis). Additionally, from Ref. [42], an expression for \( \Omega \) is determined by integrating \( \varphi \) with respect to \( \xi \),

\[ \varphi = \frac{1}{\Lambda(s+1)} \left[ \Lambda\left(\lambda + \frac{1}{2}\right)^{s+2} - \Lambda(\lambda - \xi)^{s+2} \right] \] (15)

which is applicable to Newtonian and non-Newtonian fluid flows. Associated boundary conditions are given by \( \varphi = 0 \) at \( \xi = -\frac{1}{2} \), and \( \varphi = 1 \) at \( \xi = \frac{1}{2} \).

With Newtonian fluid flow, \( s = n = 1 \), and according to Flumerfelt et al. [42], \( |\lambda| = 1/|\Lambda| \). As a result,

\[ \Omega = 1/2 + \Lambda/12 \] (16)

where, with Newtonian flow,

\[ \Lambda = \frac{h}{(R_1 + R_2)\mu/2} \frac{\Delta P}{H} \]

\[ V = (R_1 + R_2)\omega/2, \quad W = R_2 - R_1, \quad \text{and} \ \Delta \theta = \pi/2. \]

The resulting dimensional volumetric flow rate is then given by

\[ Q = \frac{c_0 h(R_2^2 - R_1^2)}{4} - \frac{h^3}{6\mu(R_1 + R_2)\Delta \theta} \Delta P \] (17)

This equation gives results which are numerically equivalent to Eq. (14), which is from Blanchard et al. [37]. The different forms of these two equations are a result of different \( z \)-coordinate origin locations, and the strict applicability of Eqs. (16) and (17) only along the mean radial line, which means that \( \Omega \) is actually given on a per unit width basis.

With non-Newtonian fluid flow, the relationship between \( \lambda \) and \( \Lambda \) depends upon the value of \( \Lambda \). Note that two different qualitative velocity profile arrangements are encountered in the present investigation. Flumerfelt et al. [42] refer to these as case I and case II types of generalized Couette flow. Case I occurs when there is no maximum or minimum in the velocity profile for \( -\frac{1}{2} \leq \xi \leq \frac{1}{2} \). With this arrangement, \( |\lambda| \leq (s + 1)^{1/4} \). Since the \( n \) values for the flow solutions of the present study are between 0.9 and 1.0, the experimental condition boundary between case I and case II occurs when \( \Lambda \) is approximately 1.8. Thus, with case I, when \( \Lambda < 1.8 \), the relationship between \( \Lambda \) and \( \lambda \) is expressed using an equation of the form

\[ |\lambda| = \frac{1}{|\Lambda|} - \frac{1}{24} |\Lambda| \] (18)

Equations (12) and (15) then give appropriate expressions for the velocity profile and the volumetric flow rate.

For the case II arrangement, \( |\lambda| \geq (s + 1)^{1/4} \), which means that \( \Lambda > 1.8 \). The relationship between \( \Lambda \) and \( \lambda \) is then given by

\[ |\lambda| = \frac{2^{s-1}}{|\Lambda|^2} \] (19)

Case II occurs when the velocity profile has a maximum or minimum for \( -\frac{1}{2} \leq \xi \leq \frac{1}{2} \). Using appropriate values of \( \Lambda \) and \( \lambda \), and the nondimensional volumetric flow rate is determined using Eq. (15).

The case II situation is always present when the overall volumetric flow rate within the VDP is zero. With this arrangement, \( Q = 0 \), and \( \Omega = 0 \). Substituting the relationship given by Eq. (19) into (15), and setting \( \Omega = 0 \) subsequently produces

\[ \frac{1}{2} - \frac{2^{s-1}}{|\Lambda|^2} = 0 \]

\[ + \frac{1}{2^{s-1}} \left[ \Lambda\left(\frac{1}{2} + |\Lambda|^2\right)^{s+2} - \Lambda\left(\frac{1}{2} + |\Lambda|^2\right)^{s+2} \right] \] (20)

Within this expression, \( \Delta P \) and \( \Lambda \) are then determined for \( Q = 0 \) using Eq. (10).

With the case II arrangement, the nondimensional velocity profile given by Eq. (12) is expressed differently for \( \xi < \lambda \) and \( \xi > \lambda \), where \( \lambda \) is the dimensionless constant of integration. These two flow regions are denoted \( \varphi^< \) and \( \varphi^> \), respectively,

\[ d\varphi^< = \Lambda\left|\lambda^< - (\lambda - \xi)^{s+1}\right| d\xi \]

\[ d\varphi^> = -\Lambda\left|\lambda^> - (\xi - \lambda)^{s+1}\right| d\xi \]

According to Flumerfelt et al. [42], integrating these expressions from \( \xi = -\frac{1}{2} \) and to a \( \xi \) location within the VDP passage flow, and applying the boundary conditions, then gives equations which have the following forms:

\[ \varphi^< = \frac{\Lambda\left|\lambda^< - (\lambda - \xi)^{s+1}\right|}{(s+1)} \] (21)

\[ \varphi^> = \frac{\Lambda\left|\lambda^> - (\xi - \lambda)^{s+1}\right|}{(s+1)} + 1 \] (22)

Additional discussion of the related nondimensional velocity profiles is provided later in the paper when experimental results are presented.

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Results and Discussion

The experimentally measured and analytically predicted data illustrate the pressure rise and flow rate characteristics of the viscous disc pump, especially as the flow behavior deviates by greater amounts from nominal Newtonian behavior, to become more non-Newtonian. All data are given for a flow passage height $h$ of 200 $\mu$m.

**Dimensional Pressure Rise Variations With Dimensional Volumetric Flow Rate for Different Disc Rotational Speeds.** The first part of this discussion considers variations of dimensional pressure rise with dimensional flow rate for the disk viscous pump with a flow passage height of 200 $\mu$m. Experimentally measured and analytically predicted data are compared for different disk rotational speeds, including the deviations which occur as the fluid within the pump becomes more non-Newtonian. Deviations from Newtonian behavior are induced by adding different concentrations of sucrose to the purified water, with increasing non-Newtonian characteristics as sucrose concentration increases from 0% (pure water) to 10% by mass. Variations of dimensional pressure rise with dimensional volumetric flow rate are illustrated by the data given in Figs. 5–8, where each data set is obtained at constant disc rotational speed. The experimental data within these four figures are obtained using pure water (with no added sucrose), 1% sucrose, 5% sucrose, and 10% sucrose concentrations as the working fluids, respectively, with dimensional rotational speeds ($\Omega$) between 1200 and 2500 rpm. As the disc rotational speed is held constant, the flow rate is varied by changing the adjustable valve on the flow meter which is shown in Fig. 4(a).

The solid lines in Fig. 5 represent analytically determined values using Eq. (15) for a pure Newtonian fluid for disc rotational speeds $\Omega$ of 1200 and 2500 rpm. These values are computed by solving Eq. (15) for $\Delta P$ as it varies with $Q$. Associated experimental data are also included within the figure. These results show a linear relationship between the dimensional pressure rise and dimensional flow rate for the VDP for each different impeller rotational speed, which is consistent with results from a variety of other macroscale viscous pumps [28–34,37]. For both values of $\Omega$, pressure rise increases as flow rate decreases, such that maximum pressure rise is present for $Q = 0$, and the maximum flow rate is present for $\Delta P = 0$. Figure 5 also shows that the experimental data are in reasonable agreement with Eq. (15) results. The slopes of the data for each rotational speed in Fig. 5 are approximately the same, which seems to suggest that the slope values associated with variations of $\Delta P$ and $Q$ may be independent of rotational speed.

According to Blanchard et al. [37], similar Newtonian flow trends of pressure rise and flow rate variations are observed for flow passage heights of 40, 73, and 246 $\mu$m. The data given in Fig. 5 also show that the single-VDP with a flow passage height

![Fig. 5 Dimensional pressure rise variation with dimensional volumetric flow rate for pure water for disc rotational speeds of 1200–2500 rpm. The chamber height of the single-disk viscous pump is 200 $\mu$m. N = Newtonian fluid analytic result; nN = non-Newtonian fluid analytic result; E = experimental data.](http://example.com/fig5)

![Fig. 6 Dimensional pressure rise variation with dimensional volumetric flow rate for water with 1.0% sucrose solution for disc rotational speeds of 1200–2500 rpm. The chamber height of the single-disk viscous pump is 200 $\mu$m. N = Newtonian fluid analytic result; nN = non-Newtonian fluid analytic result; E = experimental data.](http://example.com/fig6)

![Fig. 7 Dimensional pressure rise variation with dimensional volumetric flow rate for water with 5.0% sucrose solution for disc rotational speeds of 1200–2500 rpm. The chamber height of the single-disk viscous pump is 200 $\mu$m. N = Newtonian fluid analytic result; nN = non-Newtonian fluid analytic result; E = experimental data.](http://example.com/fig7)

![Fig. 8 Dimensional pressure rise variation with dimensional volumetric flow rate for water with 10.0% sucrose solution for disc rotational speeds of 1200–2500 rpm. The chamber height of the single-disk viscous pump is 200 $\mu$m. N = Newtonian fluid analytic result; nN = non-Newtonian fluid analytic result; E = experimental data.](http://example.com/fig8)
of 200 μm is useful for applications that require pressure rises up to approximately 200 Pa and flow rates up to 0.000000045 m³/s.

The dimensional pressure rise and dimensional volumetric flow rate experimental data in Fig. 6 are given for water with 1.0% sucrose solution for disc rotational speeds of 1200, 1800, and 2500 rpm. Analytically determined values for the same flow configuration and conditions are again determined using Eq. (15) for a pure Newtonian fluid. Overall, qualitative and quantitative trends are similar to the results in Fig. 5, with pressure variation data showing linear variations with volumetric flow rate, negative slopes of these data for each disc rotational speed, approximately invariant data slopes as disc rotational speed changes, and reasonable agreement between data and analysis for most all investigated conditions. Such characteristics provide an indication that a 1.0% sucrose solution is not large enough to induce significant non-Newtonian flow behavior.

However, in contrast, results in Figs. 7 and 8 show dramatically different behavior, with greater experimental data deviations from Newtonian behavior, as the concentration of sucrose in the water increases to 5, and then, to 10% (by mass). Note that the experimental and analytical data in these figures are presented for disc rotational speeds of 1200, 1800, and 2500 rpm. The first data trend of interest is associated with characteristics of the Newtonian fluid flows, which are based upon Eq. (15), and are provided for each disc rotational speed in Figs. 7 and 8. Compared to these data, pressure rise values which are associated with the non-Newtonian data in Figs. 7 and 8 are generally lower when compared at a particular volumetric flow rate. In addition, differences between Newtonian and non-Newtonian behavior, and associated pressure rise values, become larger as volumetric flow rate decreases, for each value of dimensional disc rotational speed Ω and sucrose concentration.

Larger deviation from Newtonian behavior is expected as local fluid shear stress levels increase. Within the present VDP, such increases are associated with larger pressure increases, and smaller volumetric flow rates, for a particular value of Ω. Note that differences between the two types of flow behavior are much more significant when the sucrose concentration is 10%, relative to the 5% situation. Another important conclusion from the results given in Figs. 7 and 8 is general agreement between the experimental data, and the non-Newtonian analytic results, which are represented by Eq. (15). Such agreement validates the shear stress model (Eq. (8)) employed to represent non-Newtonian behavior, as well as the analytic equations and tools which are employed to predict associated behavior over a range of experimental conditions.

When compared at particular values of volumetric flow rate Q and dimensional disc rotational speed Ω, lower non-Newtonian data pressure rise values, relative to Newtonian data in Figs. 7 and 8, means that local stress magnitudes are also lower. Such differences in shear stress become more significant as sucrose concentration increases and as dimensional volumetric flow rate becomes smaller. Associated shear stress magnitudes are given by Eq. (8), which illustrates dependence upon \( \left[ \frac{dh_{Q_{(z)}}}{dz} \right]^{n} \) and μ. Magnitudes of effective absolute viscosity μ are given in Table 1, which shows that values generally increase as sucrose concentration increases and as non-Newtonian influences become more important. This trend is also illustrated by the data presented in Fig. 9, which also includes variation of the power-law parameter n for the local shear stress relationship, also dependent upon sucrose concentration. Decreased magnitudes of local shear stress, which occur as non-Newtonian influences become more important (at specific values of Q and Ω), are thus a consequence of greater influences of the local velocity gradient/power law term, rather than the effective absolute viscosity term (which behaves in a manner which is opposite to experimental observations).

**Dimensional Pressure Rise Variations With Disc Rotational Speed for Zero Volumetric Flow Rate.** In many practical situations, flow rate and pressure rise application requirements are met by changing the disc rotational speed and by adjusting a flow regulating valve. The data presented previously in Figs. 5–8 show that, for low velocity gradient/power law term, rather than the effective absolute viscosity term (which behaves in a manner which is opposite to experimental observations), to illustrate case I and case II flow behavior for non-Newtonian fluid flow with a 10% sucrose solution at a rotational speed of 2500 rpm.
These experimentally measured and analytically predicted data are provided for non-Newtonian fluid behavior associated with sucrose concentrations of 5% and 10% (by mass) in Figs. 11 and 12. When the flow rate is set to zero, Eq. (20), for non-Newtonian fluid flow, shows that the pump pressure rise varies linearly with rotational speed. This analytic data trend is illustrated for two sucrose concentrations of 5% and 10% in Figs. 11 and 12, respectively. These same figures show experimental data with the same quantitative trend for the same sucrose concentrations, with a channel height \( h \) of 200 \( \mu \text{m} \) and disc rotational speeds ranging between 1200 and 2500 rpm. The linearly increasing pressure rise with rotational speed for each sucrose concentration in Figs. 11 and 12 represents the maximum pressure rise between the pressure tap locations which are shown in Fig. 2. Note that static pressure continues to increase as the flow advects from the fluid inlet port, to a location near the most-upstream pressure tap, to a location near the most-downstream pressure tap, to the fluid outlet port. Consequently, theoretical values for the maximum pressure rise \( \Delta P \) are obtained using \( \Delta \theta = \pi/2 \) in Eq. (20). Note that \( \Delta \theta = 1.067\pi \) corresponds to the angular span of the shear channel between the edges of the fluid inlet and outlet ports. According to the results which are given in Figs. 11 and 12, Eq. (20) gives a good representation of the pressure rise between the pressure tap locations for any given flow rate, rotational speed, and flow passage height.

Note that the Reynolds number for the VDP with a flow passage height of 200 \( \mu \text{m} \) is 92.5 for a rotational speed of 2500 rpm, where Reynolds number is defined as

\[
Re = \frac{\rho \omega (R_2 + R_1) h}{2\mu}
\]

One assumption made in deriving the analytic results in the previous section is that the Reynolds number is small. The data in Figs. 5–12 suggest that Eqs. (10) and (20) are valid for Re less than approximately 10\(^3\) for a flow passage height of 200 \( \mu \text{m} \), which corresponds to a rotational speed of 2500 rpm. This range of Reynolds number validity is a result of neglecting advection terms in Eqs. (1) and (2). However, note that some of these advection terms are also negligible because the flow in the passage is maintained at or near to a fully-developed flow condition. Increasing magnitudes of the advection terms are generally associated with turbulent flow development, which is also expected to occur for Re greater than approximately 10\(^3\).

### Dimensional Pressure Rise Deviations Due to Non-Newtonian Fluid Behavior

Here, magnitudes and variations of \( \Delta P^* \) are presented and discussed, as this quantity changes with dimensional volumetric flow rate. Here, \( \Delta P^* \) is the dimensional pressure rise change at a particular flow rate and sucrose concentration, as the flow changes from Newtonian to non-Newtonian behavior. These data are obtained from the results which are presented in Figs. 7 and 8. Note that both analytic and experimental data are given for different disc rotational speeds.

Figure 13 presents the pressure difference \( \Delta P^* \) for 10% sucrose solution for rotational speeds of 1200–2500 rpm. The viscous disk pump chamber height is 200 \( \mu \text{m} \). E = experimental data; nN = non-Newtonian analytic result.

Figure 14 presents the pressure difference \( \Delta P^* \) for 5% sucrose solution for rotational speeds of 1200–2500 rpm. The viscous disk pump chamber height is 200 \( \mu \text{m} \). E = experimental data; nN = non-Newtonian analytic result.
variation of $\Delta P^*$ with dimensional volumetric flow rate is approximately linear, for each sucrose solution concentration and disk rotational speed. In addition, the associated quantitative experimental data trends in Figs. 13 and 14 are in good agreement with non-Newtonian analytic results.

As sucrose concentration increases, flow behavior becomes increasingly non-Newtonian. The data in Fig. 15 are presented for a disk rotational speed of 2500 rpm, with 5% and 10% sucrose solutions, to illustrate the influences of sucrose concentration. The associated magnitudes of the pressure difference ($\Delta P^*$) show that the fluid with the larger sucrose concentration produces significantly larger differences between Newtonian and non-Newtonian fluid flow, for each value of dimensional volumetric flow rate ($Q$). As for the results in the previous two figures, the experimental data which are presented in Fig. 15 compare favorably with the non-Newtonian analytic results determined using Eq. (1).

An additional illustration of non-Newtonian influences (as sucrose concentration increases), is shown by the analytically predicted data in Fig. 16. Here, pressure difference ($\Delta P^*$) between Newtonian and non-Newtonian fluid flows is given for zero dimensional volumetric flow rate ($Q = 0$) and for different dimensional disk rotational speeds. These pressure difference data are given as dependent upon sucrose solution concentration. As the sucrose concentration increases, significant increases of $\Delta P^*$ are evident, along with larger $\Delta P^*$ variations with dimensional disk rotational speed.

Normalized Pressure Rise Deviations Due to Non-Newtonian Fluid Behavior. Figure 17 presents magnitudes of $\Delta P^*$, which are normalized using fluid density, volumetric flow rate, and disk rotational speed. These experimentally measured data are presented as they change with dimensional volumetric flow rate for different disc rotational speeds and different sucrose concentrations. Here, the normalized form of $\Delta P^*$ is denoted as $\Delta P^{**}$, and given as follows:

$$\Delta P^{**} = \Delta P^*/(\rho \omega^{4/3} Q^{2/3}) \quad (24)$$

Within this equation, $\omega = 2\pi \Omega/60$. The associated data in Fig. 17 are presented for 5% and 10% sucrose solutions, and for disk rotational speeds of 1200–2500 rpm. The data in Fig. 17 show a reasonable collapse over the range of experimental conditions which are considered, especially when compared with the data variations which are evident in previous figures. This provides evidence that VDP pressure rise deviations, from non-Newtonian flow influences, scale with $\rho$, $\omega^{4/3}$, and $Q^{2/3}$.

Non-Newtonian Influences on Pump Performance. A key parameter which characterizes pump performance is the dimensional slope of pressure rise variation with dimensional volumetric flow rate. For rotary pumps with impeller blades, radial blades with no curvature are generally associated with a zero slope magnitude. More negative values of this slope are then generally associated with blades with greater amounts of backward curvature. The slope data which are presented in Fig. 18 are presented for different sucrose concentration solutions and different dimensional disk rotational speeds from 1200 to 2500 rpm. These results are obtained from the analytical data given in Figs. 5–8, and represent a chamber height for the single-disk viscous pump of 200 $\mu$m. Because all dimensional slopes in Fig. 18 are negative, it is apparent that the VDP performance is similar to that produced by an impeller with backward curved blades. Such backward-curved similarity is present regardless of rotational speed, and regardless of whether or not non-Newtonian effects are present. In general, slopes with non-Newtonian influences are less negative compared to the Newtonian slopes, when compared at the same sucrose concentration and dimensional disk rotational speed. Note that the Newtonian data for sucrose concentrations of 1.0%, 5.0%, and 10.0% are based upon the associated analytic Newtonian data which are presented within Figs. 6–8. Figure 18 also shows that the associated data show larger divergence (between data sets with and without non-Newtonian influences) as the sucrose concentration increases, with a wider range of backward-curved simulated behavior also as sucrose concentration increases. As such,
The present investigation, which are denoted as

resulting equations which are dependent upon pressure rise

non-Newtonian fluid flows. The resulting equations are based

and sucrose concentration, as the flow changes from Newtonian to non-

Newtonian behavior. For 5% and 10% sucrose solutions at rotational

speeds of 1200–2500 rpm, the ∆P* quantity increases as the disk

dimensional rotational speed increases. In most cases, the

variation of ∆P* with dimensional volumetric flow rate is approxi-

mately linear, for each sucrose solution concentration and disk

rotational speed. Observed variations of ∆P*, representing devia-

tions due to non-Newtonian flow influences, appear to scale

approximately with ρ, cν3/2 and Q2/3.

Associated magnitudes of the pressure difference ∆P* show

that the fluid with the larger sucrose concentration (by mass)

produces significantly larger differences between Newtonian and

non-Newtonian fluid flow, for each value of dimensional volumeti-

ic flow rate Q. For each disc rotational speed, compared to New-

tonian data, ∆P dimensional pressure rise variations with

dimensional volumetric flow rate, which are associated with the

non-Newtonian data, are generally lower when compared at a par-

ticular volumetric flow rate. In addition, differences between

Newtonian and non-Newtonian behavior, and associated pressure

rise values, become larger as volumetric flow rate decreases, for

each value of dimensional disc rotational speed and sucrose con-

centration. A larger deviation from Newtonian behavior is

expected as local fluid shear stress levels increase. Within the present

VDP, such increases are associated with larger ∆P dimensional

pressure rise increases, for a particular value of dimensional disk

rotational speed. Consequently, differences between the two

types of flow behavior are much more significant when the suc-

rose concentration is 10%, relative to the 5% situation. When the

flow rate is set to zero, the dimensional pump pressure rise for

non-Newtonian fluid flow is maximum for a particular disk rota-

tional speed, and varies linearly with disk rotational speed for two

sucrose concentrations of 5% and 10%. Agreement with analytic

results, for any given flow rate, rotational speed, and flow passage

height, validates the shear stress model employed to represent

non-Newtonian behavior, as well as the analytic equations and

tools (based upon the Navier–Stokes equations) which are

employed to predict measured behavior over the investigated

range of experimental conditions.

Summary and Conclusions

A miniature viscous disk pump (VDP) is utilized to characterize

and quantify non-Newtonian fluid behavior, especially deviations

due to non-Newtonian influences relative to Newtonian flow

behavior. Deviations from Newtonian behavior are induced by

adding different concentrations of sucrose to purified water, with

increasing non-Newtonian characteristics as sucrose concentration

increases from 0% (pure water) to 10% by mass. The VDP

consists of a 10.16 mm diameter disk that rotates above a C-shaped

channel with inner and outer radii of 1.19 mm, and 2.38 mm,

respectively, and a channel depth of 200 μm. Fluid inlet and outlet

ports are located at the ends of the C-shaped channel. The advan-

tages of the present VDP compared to other micropumps and vis-

cous pumps include analytic tractability, a wide range of possible

flow rates, simplicity, constant flow rate, planar structure, well-

controlled flow rate, and the ability to pump delicate fluids with-

out disruption or damage. Within the present study, experimental
data are given for rotational speeds of 1200–2500 rpm, fluid vis-

cosities of 0.001–0.00134 Pa s, pressure rises of 0–220 Pa, and

flow rates up to approximately 0.00000005 m3/s.

The theory of Flumerfelt et al. [42] is modified and adapted for

application to the present VDP environment. Included is a new de-

velopment of expressions for dimensionless volumetric flow rate,

and normalized local circumferential velocity for Newtonian and

non-Newtonian fluid flows. The resulting equations are based

upon a power relationship between stress and strain rate, with

resulting expressions which are dependent upon pressure rise

(∆P), pump geometry (R1, R2, ∆Ω, and h), disk angular velocity

(ω), and fluid viscosity (μ). These derived expressions are applica-

tible to two different qualitative velocity profile arrangements

encountered in the present investigation, which are denoted as

case I and case II types of generalized Couette flow [42].

Analytically-predicted profiles of normalized local circumferen-

tial velocity profiles are presented which illustrate associated

behavior. When α < 1.8, case I occurs when there is no maximum

or minimum in the velocity profile for −1/2 ≤ ζ ≤ 1/2. Case II
generally occurs when α > 1.8, and the velocity profile has a maximum

or minimum for −1/2 ≤ ζ ≤ 1/2. The case II situation is always present

when the overall volumetric flow rate within the VDP is zero, as a

result of the presence of both forward and reversed, recirculating

flows within the VDP flow passage.

To quantify deviations due to the magnitude non-Newtonian

flow influences, ∆P* is employed, which represents the dimen-

sional pressure rise change at a particular flow rate and sucrose

concentration, as the flow changes from Newtonian to non-

Newtonian behavior. For 5% and 10% sucrose solutions at rota-

tional speeds of 1200–2500 rpm, the ∆P* quantity increases as the disk

dimensional rotational speed increases. In most cases, the

variation of ∆P* with dimensional volumetric flow rate is approxi-

mately linear, for each sucrose solution concentration and disk

rotational speed. Observed variations of ∆P*, representing devia-

tions due to non-Newtonian flow influences, appear to scale

approximately with ρ, cν3/2 and Q2/3.

Associated magnitudes of the pressure difference ∆P* show

that the fluid with the larger sucrose concentration (by mass)

produces significantly larger differences between Newtonian and

non-Newtonian fluid flow, for each value of dimensional volumeti-

c flow rate Q. For each disc rotational speed, compared to New-

tonian data, ∆P dimensional pressure rise variations with

dimensional volumetric flow rate, which are associated with the

non-Newtonian data, are generally lower when compared at a par-

ticular volumetric flow rate. In addition, differences between

Newtonian and non-Newtonian behavior, and associated pressure

rise values, become larger as volumetric flow rate decreases, for

each value of dimensional disc rotational speed and sucrose con-

centration. A larger deviation from Newtonian behavior is

expected as local fluid shear stress levels increase. Within the present

VDP, such increases are associated with larger ∆P dimensional

pressure rise increases, for a particular value of dimensional disk

rotational speed. Consequently, differences between the two

types of flow behavior are much more significant when the suc-

rose concentration is 10%, relative to the 5% situation. When the

flow rate is set to zero, the dimensional pump pressure rise for

non-Newtonian fluid flow is maximum for a particular disk rota-

tional speed, and varies linearly with disk rotational speed for two

sucrose concentrations of 5% and 10%. Agreement with analytic

results, for any given flow rate, rotational speed, and flow passage

height, validates the shear stress model employed to represent

non-Newtonian behavior, as well as the analytic equations and

tools (based upon the Navier–Stokes equations) which are

employed to predict measured behavior over the investigated

range of experimental conditions.

Nomenclature

h = flow passage height of the disk pump
n = power-law coefficient for non-Newtonian stress-strain

relationship
p = local static pressure
P = local static pressure
∆P = static pressure rise between pressure ports 1 and 2,

p2 − p1

∆P_out-in = static pressure rise between fluid inlet and outlet ports

∆P* = pressure difference between Newtonian and non-

Newtonian flow at a particular volumetric flow rate

and sucrose concentration

∆P** = nondimensional pressure rise

Q = dimensional volumetric flow rate
r = radial coordinate
R1 = inner radius of the pump chamber
R2 = outer radius of the pump chamber
s = 1/h
v0 = local circumferential component fluid velocity
v1 = local radial component fluid velocity
v2 = local normal component fluid velocity
V = r0
W = channel width
z = normal coordinate

Greek Symbols

Λ = parameter given by Eq. (10)
ζ = z/h
\[ \mu = \text{absolute viscosity, effective absolute viscosity} \]
\[ \lambda = \text{dimensionless constant of integration} \]
\[ \rho = \text{fluid density} \]
\[ \tau = \text{local shear stress} \]
\[ \phi = \nu_0 V \]
\[ \omega = \text{rotational speed of the disk, } 2\pi \Omega / 60 \]
\[ \Omega = \text{dimensional rotational speed of the disk} \]
\[ \theta = \text{circular coordinate} \]
\[ \Delta \theta = \text{circular span between two angular locations} \]

References